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A Chebyshev Method for the Solution of Boundary Value Problems

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An expansion procedure using the Chebyshev polynomials as base functions is proposed. The method yields more accurate results than either of the Galerkin or tau methods as indicated from solving the Orr–Sommerfeld equation for both the plane Poiseuille flow and the Blasius velocity profile. The Chebyshev approximation is also applied to resolve the radial dependence of the flow field for a circular cylinder or a sphere in a uniform flow.

1. INTRODUCTION

The use of Chebyshev polynomial approximations for the solution of boundary value problems in fluid mechanics has been advocated and developed by Orszag [1]. These approximations, which include the Galerkin and tau spectral methods and the pseudospectral collocation method, are discussed in detail by Gottlieb and Orszag [2]. In this paper we propose a spectral procedure based on the Galerkin method which employs the Chebyshev polynomials and yields more accurate results than the tau method for the same computer requirement. Moreover, this expansion procedure is capable of handling nonhomogeneous boundary conditions directly.

In Section 2 we solve the Orr–Sommerfeld equation. Here we introduce the representation procedure and the necessary recurrence relations which are listed in the Appendix. We also show how this spectral method may be applied to boundary value problems by solving the Blasius equation. Comparison of the eigenvalues for either of the plane Poiseuille flow or the Blasius velocity profile with those computed by Orszag [3] and Grosch and Orszag [4] using the tau method indicates that we obtain higher accuracy with the same number of expansion coefficients.

The solution of the Navier–Stokes equations for a circular cylinder in a uniform flow of an incompressible fluid is considered in Section 3. A general formulation for the problem is presented and the steady two-dimensional flow is produced by a sine series (Dennis and Chang [5]) representation in the azimuthal direction and a Chebyshev approximation in the radial direction. The analogous case of a sphere in a uniform flow is considered in Section 4. For the steady axisymmetric flow field we use a Legendre series (Dennis and Walker [6]) in the latitudinal direction and the Chebyshev representation in the radial direction. Thus, in both cases, the problem

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was reduced to a nonlinear algebraic system of equations in the expansion coefficients which were solved by Newton iteration. We were able to produce accurate solutions for values of the Reynolds number less than those required for laminar separation in either case. Because all computations were performed on a VAX-11/780 computer we did not compute accurate solutions at higher values of the Reynolds numbers as this would require impractical computer time. However, in the conclusion we discuss how the observed instabilities of these restricted classes of solutions at higher Reynolds numbers may be predicted by linear stability methods.

2. Orr–Sommerfeld Equation

Linear stability analysis of an incompressible parallel shear flow $U(z)$, $-1 \leqslant z \leqslant 1$, to two-dimensional perturbations of the form

$$\psi(z)\, e^{i\alpha(x-ct)}$$

leads to the Orr–Sommerfeld equation

$$\psi^{(4)} - 2\alpha^2\psi^{(2)} + \alpha^4\psi - i\alpha R\left[(U-c)(\psi^{(2)} - \alpha^2\psi) - U^{(2)}\psi\right] = 0, \tag{2.1a}$$

$$\psi = \psi^{(1)} = 0 \qquad \text{at} \quad z = \pm 1, \tag{2.1b}$$

where α and c are the disturbance wavenumber and complex wave speed, respectively. All quantities are dimensionless and R is the Reynolds number. Superscript numbers in parenthesis on functions of z indicate derivatives with respect to z.

In the Galerkin or tau methods one assumes a representation for $\psi(z)$ in terms of Chebyshev polynomials $T_n(z)$. The representations of the derivatives of $\psi(z)$ are then obtained by use of the identities [3]

$$2T_n = \frac{C_n}{n+1} T^{(1)}_{n+1} - \frac{d_{n-2}}{n-1} T^{(1)}_{n-1}, \tag{2.2a}$$

$$2zT_n = C_n T_{n+1} + d_{n-1} T_{n-1}, \tag{2.2b}$$

where $C_n = d_n = 0$ if $n < 0$, $C_0 = 2$, $d_0 = 1$, $C_n = d_n = 1$ if $n > 0$. Here we assume a representation for the highest derivative of ψ in (2.1) of the form

$$\psi^{(4)} = \sum_{j=0}^{N} a_j T_j. \tag{2.3a}$$

The representations of the lower derivatives are found by successive integration and use of (2.2)

$$\psi^{(3)} = \sum_{j=0}^{N+1} \sum_{i=0}^{N} f^{(3)}_{ji} a_i T_j + C_1 T_0, \tag{2.3b}$$

$$\psi^{(2)} = \sum_{j=0}^{N+2} \sum_{i=0}^{N} f_{ji}^{(2)} a_i T_j + C_1 T_1 + C_2 T_0, \tag{2.3c}$$

$$\psi^{(1)} = \sum_{j=0}^{N+3} \sum_{i=0}^{N} f_{ji}^{(1)} a_i T_j + \frac{C_1}{4} T_2 + C_2 T_1 + C_3 T_0, \tag{2.3d}$$

$$\psi = \sum_{j=0}^{N+4} \sum_{i=0}^{N} f_{ji}^{(0)} a_i T_j + \frac{C_1}{24} T_3 + \frac{C_2}{4} T_2 + \left(C_3 - \frac{C_1}{8}\right) T_1 + C_4 T_0. \tag{2.3e}$$

The constants $f_{ji}^{(\beta)}$, $\beta = 0, 1, 2, 3$, are listed in the Appendix, and the constants of integration C_1, C_2, C_3, and C_4 are determined in terms of the expansion coefficients a_i so that the boundary conditions (2.1b) are satisfied, we find

$$C_1 = \sum_{i=0}^{N} \tfrac{3}{2}[(\sigma_i^{(0)} - \bar{\sigma}_i^{(0)}) - (\sigma_i^{(1)} + \bar{\sigma}_i^{(1)})]\, a_i, \tag{2.4a}$$

$$C_2 = \sum_{i=0}^{N} -\tfrac{1}{2}[\sigma_i^{(1)} - \bar{\sigma}_i^{(1)}]\, a_i, \tag{2.4b}$$

$$C_3 = \sum_{i=0}^{N} -\tfrac{1}{8}[3(\sigma_i^{(0)} - \bar{\sigma}_i^{(0)}) + (\sigma_i^{(1)} + \sigma_i^{(1)})]\, a_i, \tag{2.4c}$$

$$C_4 = \sum_{i=0}^{N} \tfrac{1}{2}[-(\sigma_i^{(0)} + \bar{\sigma}_i^{(0)}) + \tfrac{1}{4}(\sigma_i^{(1)} - \bar{\sigma}_i^{(1)})]\, a_i, \tag{2.4d}$$

where

$$\sigma_i^{(\beta)} = \sum_{j=0}^{N+4-\beta} f_{ji}^{(\beta)}, \qquad \bar{\sigma}_i^{(\beta)} = \sum_{j=0}^{N+4-\beta} (-1)^j f_{ji}^{(\beta)}, \qquad \beta = 0, 1, 2, 3. \tag{2.4e}$$

Upon substitution of (2.4a)–(2.4d) into (2.3b)–(2.3e) we may conveniently rewrite the expansions (2.3b)–(2.3e) in the forms

$$\psi^{(\beta)} = \sum_{j=0}^{N+4-\beta} \sum_{i=0}^{N} g_{ji}^{(\beta)} a_i T_j, \qquad \beta = 0, 1, 2, 3, \tag{2.5a}$$

where, for example,

$$g_{ji}^{(3)} = f_{ji}^{(3)} + \delta_{j,0}\, \tfrac{3}{2}[(\sigma_i^{(0)} - \bar{\sigma}_i^{(0)}) - (\sigma_i^{(1)} + \bar{\sigma}_i^{(1)})], \tag{2.5b}$$

and with similar expressions for $g_{ji}^{(\beta)}$, $\beta = 0, 1, 2$. The idea of assuming a representation for the derivative rather than for the function has been used (Jeffreys and Jeffreys [7, p. 441]) to examine term-by-term differentiation of a Fourier series for the function at a point of discontinuity of the derivative. However, in addition to uniform convergence [1], our motivation for the scheme in (2.3) is twofold: to obtain an $N + 4$ truncation for ψ using only N expansion coefficients thus higher accuracy than a corresponding tau expansion; and simultaneously satisfying all the boundary conditions on ψ and its derivatives. We also assume that the basic flow is given by

$$U^{(\beta)}(z) = \sum_{n=0}^{N_b+2-\beta} b_n^{(\beta)} T_n(z), \qquad \beta = 0, 1, 2. \tag{2.6}$$

The usual Galerkin procedure then leads to the matrix eigenvalue problem

$$\begin{aligned}\sum_{k=0}^{N} \Bigg[\frac{1}{i\alpha R} (\delta_{ik} - 2\alpha^2 g_{ik}^{(2)} + \alpha^4 g_{ik}^{(0)}) - \sum_{l=0}^{N_b+2} \sum_{j=0}^{N+2} \beta_{ilj} b_l^{(0)} g_{jk}^{(2)} \\ + \alpha^2 \sum_{l=0}^{N_b+2} \sum_{j=0}^{N+4} \beta_{ilj} b_l^{(0)} g_{jk}^{(0)} + \sum_{l=0}^{N_b} \sum_{j=0}^{N+4} \beta_{ilj} b_l^{(2)} g_{jk}^{(0)} \Bigg] a_k \\ = -c \sum_{k=0}^{N} [g_{ik}^{(2)} - \alpha^2 g_{ik}^{(0)}] a_k, \qquad i = 0, 1, \ldots, N,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\beta_{ilj} &= \frac{2}{\pi c_i} \int_{-1}^{+1} (1 - z^2)^{-1/2} T_i T_l T_j \, dz \\ &= \frac{1}{2} \left[\delta_{i,l+j} + \frac{1}{c_i} (\delta_{i,l-j} + \delta_{i,j-l}) \right],\end{aligned} \tag{2.8}$$

which follows from the identity

$$2T_j T_l = T_{j+l} + T_{|j-l|}.$$

Numerical solution of the matrix eigenvalue problem (2.7) is obtained using the IMSL routine EIGZC.

2.1. *Plane Poiseuille Flow*

Here $U(z) = 1 - z^2$. It follows that in (2.6) we have

$$N_b = 1, \qquad b_n^{(0)} = (\tfrac{1}{2}, 0, -\tfrac{1}{2}), \qquad \text{and} \qquad b_0^{(2)} = -2.$$

For the plane Poiseuille flow, the eigenfunctions are either symmetric or asymmetric about $z = 0$. In Table I we show the most unstable eigenvalue (which corresponds to a symmetric eigenfunction) computed here and in [3] for $\alpha = 1$ and $R = 10{,}000$. This shows that the tau method requires solution of a 29×29 algebraic system while we need only solve a 25×25 system to converge to the "exact value" $0.23752649 + 0.00373967i$ [3]. Although this may seem a minor gain, it is important when solving partial differential systems. Not unlike other methods [2] of computing the eigenvalues of the Orr–Sommerfeld equation we also find two spurious eigensolutions with one eigenfunction symmetric and the other asymmetric about $z = 0$.

TABLE I

The Most Unstable Eigenvalue of Plane Poiseuille Flow for $\alpha = 1$, $R = 10000$

M^a	Orszag [3]	Present Results
14	$0.23713751 + 0.00563644i$	$0.23757046 + 0.00374610i$
15	$0.23690887 + 0.00365516i$	$0.23743315 + 0.00372248i$
17	$0.23743315 + 0.00372248i$	$0.23752392 + 0.00375328i$
20	$0.23752676 + 0.00373427i$	$0.23752595 + 0.00373908i$
23	$0.23752670 + 0.00373982i$	$0.23752651 + 0.00373968i$
24		$0.23752648 + 0.00373967i$
25		$0.23752649 + 0.00373967i$
26	$0.23752648 + 0.00373967i$	$0.23752649 + 0.00373967i$
29	$0.23752649 + 0.00373967i$	$0.23752649 + 0.00373967i$

[a] M is the number of expansion coefficients.

2.2. *Blasius Velocity Profile*

In order to represent the basic flow in the form (2.6) we first consider the Blasius equation for $f(\eta)$

$$f''' + ff'' = 0, \tag{2.9a}$$

$$f(0) = f'(0) = 0, \qquad f'(\eta \to \infty) \sim 1. \tag{2.9b}$$

We will solve (2.9) by the Chebyshev method. The infinity condition will be invoked at a finite value of $\eta = \eta_e$. First, let

$$\frac{\eta}{\eta_e} = (z+1)/2, \qquad -1 \leqslant z \leqslant 1. \tag{2.10}$$

then $f(z)$ satisfies

$$f^{(3)} + \frac{\eta_e}{2} ff^{(2)} = 0, \tag{2.11a}$$

$$f(-1) = f^{(1)}(-1) = 0, \qquad f^{(1)}(1) = \eta_e/2. \tag{2.11b}$$

Now we assume the representation

$$f^{(3)} = \sum_{j=0}^{N_b}{}' c_j T_j. \tag{2.12a}$$

Integrating (2.12a) and invoking the boundary conditions (2.11b), we have

$$f^{(\beta)} = \sum_{j=0}^{N_b+3-\beta}{}' \sum_{i=0}^{N_b}{}' h_{ji}^{(\beta)} c_i T_j + \sum_{j=0}^{2}{}' d_j^{(\beta)} T_j, \qquad \beta = 0, 1, 2, \tag{2.13}$$

where $h_{ji}^{(\beta)}$ are the constants $f_{ji}^{(\beta+1)}$ modified to satisfy the homogeneous version of the boundary conditions (2.11b) in a manner similar to that leading to (2.5) and $d_j^{(\beta)}$

TABLE II

The Most Unstable Eigenvalue of the Blasius Profile for $\alpha = 0.179$, $R = 580$

η_e	$N-1$	Grosch and Orszag [4]	Present Results
10	36		$0.367590 + 0.006429i$
	44	$0.378877 + 0.000250i$	$0.367591 + 0.006429i$
20	36		$0.364034 + 0.007920i$
	44	$0.364557 + 0.007773i$	$0.364143 + 0.007959i$
	46	$0.364551 + 0.007781i$	$0.364137 + 0.007934i$
30	40		$0.362681 + 0.009052i$
	42		$0.363419 + 0.007859i$
	44	$0.363996 + 0.007888i$	$0.363997 + 0.007903i$

depend only on η_e. The usual Galerkin procedure applied to (2.11a) reduces the problem to a nonlinear algebraic system in the expansion coefficients c_j which we solve by Newton iteration. It should be noted that the basic flow $U(z)$ in (2.1a) is related to $f(z)$ by

$$U(z) = f^{(1)}(z),$$

so that $b_l^{(\beta)}$ in (2.7) are determined in terms of c_j. In Table II we list the single unstable eigenvalue for the case $R = 580$ and $\alpha = 0.179$ (because of the transformation (2.10) the values of α and R used to solve (2.7) are $0.179\eta_e/\sqrt{2}$ and $580\sqrt{2}$, respectively) as well as those of [4] for values of $\eta_e = 10$, 20, and 30 (the corresponding values of N_b used are 20, 30, and 40, respectively). We emphasize that both our results and those of [4] in Table II represent the solution of the same differential system with identical treatment of the infinity boundary conditions. However, while we solve the Blasius equation by a spectral procedure, Grosch and Orszag [4] use collocation. Again, we better approximate the "exact value" $0.36412286 + 0.00795972i$ than the tau method, in particular, with $\eta_e = 10$. This time we always find two spurious eigensolutions.

3. Circular Cylinder in Uniform Flow

The nondimensional Navier–Stokes equations are

$$\nabla \cdot \mathbf{V} = 0 \tag{3.1a}$$

$$\frac{\partial}{\partial t}\mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \frac{2}{R}\nabla^2\mathbf{V}, \tag{3.1b}$$

where the Reynolds number $R = 2Ub/\nu$, and b is the radius of a cylinder in a uniform stream U of an incompressible fluid with kinematic viscosity ν. The motion is referred to the cylinderical coordinate system (r, θ, y) of Fig. 1. The boundary conditions are

$$\mathbf{V}(1, \theta, y, t) = \mathbf{0}, \tag{3.2a}$$

$$\mathbf{V}(r, \theta, y, t) = \mathbf{V}(r, \theta + 2\pi, y, t), \tag{3.2b}$$

$$\mathbf{V}(r \to \infty, \theta, y, t) \sim (-\cos\theta, \sin\theta, 0). \tag{3.2c}$$

Taking the curl of (3.1b) twice and using (3.1a) leads to

$$\frac{\partial}{\partial t}\boldsymbol{\omega} = \frac{2}{R}\nabla^2\boldsymbol{\omega} + \nabla \times (\mathbf{V} \times \boldsymbol{\omega}), \tag{3.3a}$$

$$\frac{\partial}{\partial t}\nabla^2\mathbf{V} = \frac{2}{R}\nabla^4\mathbf{V} - \nabla \times [\nabla \times (\mathbf{V} \times \boldsymbol{\omega}), \tag{3.3b}$$

where the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{V}$. The y component of (3.3a), (3.3b) are

$$\frac{\partial}{\partial t}\omega_y = \frac{2}{R}\nabla^2\omega_y + \hat{e}_y \cdot \nabla \times (\mathbf{V} \times \boldsymbol{\omega}), \tag{3.4a}$$

$$\frac{\partial}{\partial t}\nabla^2 V_y = \frac{2}{R}\nabla^4 V_y - \hat{e}_y \cdot \nabla \times [\nabla \times (\mathbf{V} \times \boldsymbol{\omega})], \tag{3.4b}$$

where subscripts denote components and $\hat{e}_y$ is a unit vector in the y direction. We represent the solenoidal vector velocity field in terms of two scalar functions Φ and Ψ (Chandrasekhar [8, p. 24]) as

$$\mathbf{V} = \nabla \times (\Psi\hat{e}_y) + \nabla \times (\nabla \times \Phi\hat{e}_y), \tag{3.5}$$

If we look for two-dimensional solutions

$$\frac{\partial}{\partial y} = 0, \qquad V_y = 0,$$

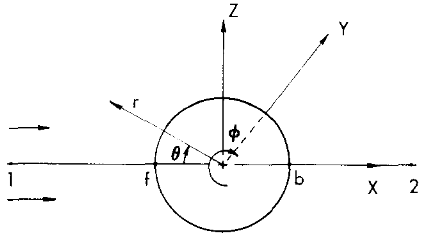

FIG. 1. The cylindrical (r, θ, y) and spherical (r, θ, ϕ) coordinate systems.

then $\Phi \equiv 0$ and $\Psi(r, \theta, t)$ is simply the streamfunction, and we need only solve (3.4a). The basic steady flow is assumed to have the symmetry given by

$$\Psi(r, \theta) = -\Psi(r, -\theta). \tag{3.6}$$

The general formulation of the problem in (3.4) and (3.5) would allow one to study the instability of the basic flow in (3.6) to different classes of perturbations.

The Fourier sine series representation [5] for $\Psi(r, \theta)$

$$\Psi = \sum_{n=1}^{N_\theta} f_n(r) \sqrt{2/\pi} \sin n\theta, \tag{3.7}$$

is used in (3.4a) to yield

$$\frac{2}{R} D_l^2(f_l) = \sum_{n=1}^{N_\theta} \sum_{m=1}^{N_\theta} A_{nlm} \frac{f_n(D_m f_m)'}{r} - A_{mln} \frac{f'_n(D_m f_m)}{r}, \tag{3.8a}$$

where

$$D_l = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{l^2}{r^2}, \tag{3.8b}$$

$$A_{nlm} = \frac{n}{\sqrt{2\pi}} (\delta_{n,|m-l|} - \delta_{n,l+m}), \tag{3.8c}$$

and primes are derivatives with respect to r. The boundary conditions satisfied by $f_n(r)$, from (3.2a), (3.2c), (3.5), and (3.7) are

$$f_n = f'_n = 0 \qquad \text{at} \quad r = 1, \tag{3.9a}$$

$$f_n \sim -\sqrt{\pi/2}\, r\delta_{1n} \qquad \text{as} \quad r \to \infty. \tag{3.9b}$$

We will solve (3.8), (3.9) by replacing the infinite range of r with a finite one. For the boundary conditions at infinity we invoke the least restrictive or soft boundary conditions (see Fornberg [9])

$$f'_n = -\sqrt{\pi/2}\, \delta_{1n} \qquad \text{and} \qquad f'''_n = 0 \quad \text{at } r = e^a, \tag{3.9c}$$

for some value of a. These boundary conditions may be deduced from the asymptotic form (Underwood [10])

$$f_n \sim a_n r^{2-n} + b_n r^{-n}, \tag{3.9d}$$

which is valid far from the body. The transformation

$$r = e^{(a/2)(z+1)}, \qquad -1 \leqslant z \leqslant 1, \tag{3.10}$$

TABLE III

Some Properties of the Solution for a Cylinder in a Uniform Flow

	Dennis and Chang [5]			Present Results[a]				
R	c_d	$P_f - P_1$	$P_b - P_1$	c_d	$P_f - P_1$	$P_b - P_1$	$P_b - P_2$	$P_1 - P_2$
1				10.693	1.988	−1.368	−1.396	0.003
2				6.855	1.372	−0.825	−0.818	0.001
5	4.116	0.936	−0.522	4.103	0.918	−0.595	−0.537	0.000

[a] The truncations N_r, N_θ, and VAX-11 computer time per iteration for $R = 1, 2, 5$ are, respectively, 19, 10, 7 min, 25, 14, 34 min, and 28, 20, 140 min.

is applied to (3.8) and for $f_l(z)$ we assume

$$f_l^{(4)}(z) = \sum_{j=0}^{N_r} a_{lj} T_j(z). \tag{3.11a}$$

Integration of (3.11a) and applying the boundary conditions (3.9a), (3.9c) leads to the representations

$$f_l^{(\beta)}(z) = \sum_{j=0}^{N_r+4-\beta} \sum_{i=0}^{N_r} g_{ji}^{(\beta)} a_{li} T_j + \delta_{1l} \sum_{j=0}^{3} b_j^{(\beta)} T_j, \qquad \beta = 0, 1, 2, 3, \tag{3.11b}$$

where $g_{ji}^{(\beta)}$ are $f_{ji}^{(\beta)}$ modified to satisfy the homogeneous version of (3.9a), (3.9c) and $b_j^{(\beta)}$ are functions only of the transformation parameter a in (3.10). We now use (3.11), (3.8), and (3.10) to derive a nonlinear algebraic system for a_{li} of the form

$$\sum_{m=1}^{N_\theta} \sum_{j=0}^{N_r} L_{limj} a_{mj} + B_{li}$$
$$= \sum_{m=1}^{N_\theta} \sum_{j=0}^{N_r} \sum_{n=1}^{N_\theta} \sum_{k=0}^{N_r} N_{limjnk} a_{mj} a_{nk}, \qquad l = 1, \ldots, N_\theta, \quad i = 0, 1, \ldots, N_r, \tag{3.12}$$

where the constants L, B, and N in (3.12) include contributions from $g_{ji}^{(\beta)}$ and $b_j^{(\beta)}$ in (3.11b) and the Reynolds number R. The solution to (3.12) is obtained by Newton method in about 3–5 iterations depending on the initial guess.

In Table III we show some properties of the solutions as well as computer time requirement at different resolutions N_r and N_θ. The results given are for $a = 5$ which we find satisfactory by recomputing the flow fields at $a = 5.1$. Here c_d is the drag coefficient and the pressure differences $P_f - P_1$ and $P_b - P_f$ are obtained by integration of (3.1b) along the contours $\theta = 0$ and $r = 1$. The pressure differences $P_b - P_2$ and $P_1 - P_2$ are obtained by integration along $\theta = \pi$ and $r = e^a$. The points 1, f, b, and 2 are as indicated in Fig. 1. We find that the best measure of the accuracy of the representation is the value of P_b as obtained by the two integration procedures. The agreement between our results at $R=5$ with those computed by Dennis and Chang [5] at the same value of N_θ is reasonable, the differences are attributed to the treatment of the infinity boundary conditions.

4. Sphere in a Uniform Flow

The motion is referred to the spherical coordinate system (r, θ, ϕ) of Fig. 1. The equations of motion are (3.3) and in addition to the boundary conditions (3.2a), (3.2c) we also require that $\mathbf{V}$ be finite at $\theta = 0, \pi$. The operation $\mathbf{r}\cdot$ applied to (3.3) gives

$$\frac{\partial}{\partial t}(r\omega_r) = \frac{2}{R}\nabla^2(r\omega_r) + \mathbf{r}\cdot\nabla\times(\mathbf{V}\times\boldsymbol{\omega}), \tag{4.1a}$$

$$\frac{\partial}{\partial t}(\nabla^2 rV_r) = \frac{2}{R}\nabla^4(rV_r) - \mathbf{r}\cdot\nabla\times[\nabla\times(\mathbf{V}\times\boldsymbol{\omega})]. \tag{4.1b}$$

We represent the solenoidal vector velocity field in terms of a poloidal scalar S and a toroidal scalar T [8, Appendix III] as

$$\mathbf{V} = \nabla\times(T\hat{e}_r) + \nabla\times(\nabla\times S\hat{e}_r), \tag{4.2}$$

where $\hat{e}_r$ is a unit vector in the radial direction. For axisymmetric motions

$$\frac{\partial}{\partial\phi} = 0, \qquad V_\phi = 0,$$

so that $T \equiv 0$ and $-\sin\theta(\partial S/\partial\theta)$ is a streamfunction and we need only solve (4.1b). The general formulation of the problem in (4.1), (4.2) would allow one to study the instability of a basic steady, axisymmetric solution. For such a basic state the representation [6]

$$S(r,\theta) = \sum_{n=1}^{N_\theta} rf_n(r)\,P_n(\cos\theta), \tag{4.3}$$

where $P_n(x)$ are normalized Legendre polynomials, is used in (4.1b) to give

$$\frac{2}{R}D_l^2(f_l) = \sum_{n=1}^{N_\theta}\sum_{m=1}^{N_\theta} C_{nlm}\frac{(f_n D_m f_m)'}{r} + C_{lnm}\frac{(rf_n)'\,D_m f_m}{r^2}, \tag{4.4a}$$

where

$$D_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} \tag{4.4b}$$

TABLE IV

Some Properties of the Solution for a Sphere in a Uniform Flow

	Dennis and Walker [6]			Present Results[a]			
R	c_d	$P_f - P_1$	$P_b - P_1$	c_d	$P_f - P_1$	$P_b - P_1$	$P_b - P_2$
1	13.72	3.876	−3.008	13.78	3.877	−3.037	−2.991
5	3.605	1.299	−0.601	3.617	1.282	−0.652	−0.562
10	2.212	0.939	−0.327	2.207	0.900	−0.378	−0.244

[a] The truncations N_r, N_θ, and VAX-11 computer times in min/iteration are 15, 6, 2; 19, 14, 40; and 23, 16, 105 for $R = 1, 5, 10$, respectively.

and

$$C_{nlm} = \frac{1}{2}\frac{n(n+1)}{l(l+1)}\left[l(l+1) + m(m+1) - n(n+1)\right]\int_{-1}^{+1} P_n P_l P_m \, dx. \tag{4.4c}$$

As in the case of the cylinder (3.9a), (3.9c) the boundary conditions we use are

$$f_l = f_l' = 0 \qquad \text{at } r = 1, \tag{4.5a}$$

$$f_l' = \frac{-1}{\sqrt{6}}\delta_{1l}, \qquad f_l''' = 0 \quad \text{at } r = e^a. \tag{4.5b}$$

Equations (4.4), (4.5) are solved in exactly the same way we solved (3.8), (3.9a), (3.9c). We use the transformation (3.10) and expansions similar to (3.11) and finally arrive at a nonlinear algebraic system similar to (3.12).

The numerical results for the sphere are given in Table IV for $a = 5$. In this case we need more computer time than we did for the cylinder since there are more nonlinear contributions to compute (compare C in (4.4c) with A in (3.8c)). Also shown are the results of Dennis and Walker [6] computed at the same values of N_θ. As was the case for the cylinder we find that the overall accuracy of the representation may be inferred from the value of P_b as obtained by different integration procedures. Note that $P_2 - P_1$ is less than 10^{-4} for all the results presented. We should also add that the values of the drag coefficient for the cylinder or the sphere can be reasonably determined using smaller truncations than those shown in Tables III and IV.

5. Conclusion

We have presented an expansion procedure which employs the Chebyshev polynomials as base functions. Because we assume a representation for the highest derivative of the dependent variable we are able to obtain higher accuracy than the tau method (for the same computer requirement) as evidenced by the solution of the Orr–Sommerfeld equation. Application of the Chebyshev approximation to the resolution of the radial direction of the flow field of a cylinder or a sphere in a uniform flow is also presented. Although it is difficult to compare the economy in computer requirement as well as accuracy with other numerical methods, semianalytic representations of a flow field have an important advantage; linear stability methods can be applied in a straightforward manner to investigate the experimentally observed instabilities. For the cylinder, we may linearize the scalars Φ and Ψ about the basic solutions in (3.7). Likewise, for the sphere we may linearize the scalars S and T about the basic solution in (4.3). One may then restrict the stability analysis to different classes of disturbances, this is the usual approach in studying cylindrical convective systems (Charlson and Sani [11]) and spherical systems (Zebib, Schubert, and Straus [12]). However, we must be able to produce accurate basic solutions for values of the Reynolds number about 40 for the cylinder and 170 for the sphere before this stability analysis can be performed.

APPENDIX

In the Appendix we list the expressions for $f_{ji}^{(\beta)}$, $\beta = 0, 1, 2, 3$, which follow from integrating (2.3a) and using (2.2) without any constants of integration.

$$f_{ji}^{(3)} = \delta_{j,i+1}\beta_i^{(3)} + \delta_{j,i-1}\gamma_i^{(3)},$$

$$f_{ji}^{(2)} = \delta_{j,i+2}\beta_i^{(2)} + \delta_{j,i}\gamma_i^{(2)} + \delta_{j,i-2}\delta_i^{(2)},$$

$$f_{ji}^{(1)} = \delta_{j,i+3}\beta_i^{(1)} + \delta_{j,i+1}\gamma_i^{(1)} + \delta_{j,i-1}\delta_i^{(1)} + \delta_{j,i-3}\varepsilon_i^{(1)},$$

$$f_{ji}^{(0)} = \delta_{j,i+4}\beta_i^{(0)} + \delta_{j,i+2}\gamma_i^{(0)} + \delta_{j,i}\delta_i^{(0)} + \delta_{j,i-2}\varepsilon_i^{(0)} + \delta_{j,i-4}\eta_i^{(0)},$$

where $\delta_{j,i}$ is the Kronecker delta and the nonzero values of β_i, γ_i, δ_i, ε_i, and η_i (for $i \geqslant 0$ only) are

$$\beta_i^{(3)} = \frac{c_i}{2(i+1)},$$

$$\gamma_i^{(3)} = \frac{-1}{2(i-1)}, \quad i \geqslant 2,$$

$$\delta_i^{(2)} = \frac{-\gamma_i^{(3)}}{2(i-2)}, \quad i \geqslant 3,$$

$$\varepsilon_i^{(1)} = \frac{-\delta_i^{(2)}}{2(i-3)}, \quad i \geqslant 4,$$

$$\eta_i^{(0)} = \frac{-\varepsilon_i^{(1)}}{2(i-4)}, \quad i \geqslant 5.$$

$$\beta_i^{(\beta-1)} = \frac{\beta_i^{(\beta)}}{2(i+5-\beta)}, \quad \beta = 3, 2, 1, \quad i \geqslant 0,$$

$$\gamma_i^{(\beta-1)} = \frac{-\beta_i^{(\beta)} + \gamma_i^{(\beta)}}{2(i+3-\beta)}, \quad \beta = 3, 2, 1, \quad i \geqslant \beta - 2,$$

$$\delta_i^{(\beta-1)} = \frac{-\gamma_i^{(\beta)} + \delta_i^{(\beta)}}{2(i+1-\beta)}, \quad \beta = 2, 1, \quad i \geqslant \beta,$$

$$\varepsilon_i^{(0)} = \frac{-\delta_i^{(1)} + \varepsilon_i^{(1)}}{2(i-2)}, \quad i \geqslant 3,$$

These expressions avoid large roundoff errors in a manner similar to that recommended in [2, p. 118]. The procedure can, of course, be extended to higher order systems.

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